Change point models and conditionally pure birth processes; an inequality on the stochastic intensity.

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Abstract

We analyze several aspects of a class of simple counting processes, that can emerge in some fields of applications where the presence of a change-point occurs. Under simple conditions we, in particular, prove a significant inequality for the stochastic intensity.

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1 Introduction

In this note we consider the *change-point* model, described as follows. Let a random time U and a simple counting process $\{N_t\}_{t\geq 0}$ be defined on a same probability space. Let $T_1 \leq T_2 \leq \dots$ be the arrival times of $\{N_t\}_{t\geq 0}$ and denote by h_t an "history", observed in the time-interval $[0,t]$, such as

$$h_t \equiv \{T_1 = t_1, \dots, T_k = t_k, T_{k+1} > t\}$$

with $0 \equiv t_0 < t_1 < t_2 < \dots < t_k < t$.

We think of U as a change-point for $\{N_t\}_{t\geq 0}$; more precisely we assume that the latter admits an intensity, described by

$$\lim_{\Delta\to 0^+} \frac{1}{\Delta} P(T_{k+1} < t + \Delta \,|\, h_t;\, U \leq t) = \lambda_1(k) \tag{1}$$

$$\lim_{\Delta\to 0^+} \frac{1}{\Delta} P(T_{k+1} < t + \Delta \,|\, h_t;\, U > t) = \lambda_0(k), \tag{2}$$

where $\lambda_0(0), \lambda_0(1), \ldots$ and $\lambda_1(0), \lambda_1(1), \ldots$ are two given sequences of positive constants; i.e., given the observation of the history h_t, $\lambda_1(\cdot)$ would be the intensity, conditional on the knowledge that time t is after the change-point U and $\lambda_0(\cdot)$ would be the intensity, conditional on the knowledge that time t is before U. We think of the case when the random variable U is not observable (obviously U takes values in the interval $(0, \infty)$); its distribution function will be denoted by G.

More formally, let $X_t \equiv 1_{\{U<t\}}$ and denote by $\Im \equiv \{\mathcal{F}_t^{(N)}\}_{t\geq 0}$ and $\aleph \equiv \{\mathcal{F}_t^{(N,X)}\}_{t\geq 0}$ the filtrations respectively generated by the processes $\{N_t\}_{t\geq 0}$ and $\{N_t, X_t\}_{t\geq 0}$; we are then assuming that the stochastic intensity of $\{N_t\}_{t\geq 0}$ with respect to $\aleph$ is

$$\mu_t^{\aleph} = \lambda_1(N_t)X_t + \lambda_0(N_t)(1 - X_t).$$

The intensity of $\{N_t\}_{t\geq 0}$ w.r.t. the "internal" filtration $\Im$ is then specified by the position:

$$\mu_t^{\Im} = \lambda_1(N_t)P(U \leq t|\mathcal{F}_t^{(N)}) + \lambda_0(N_t)P(U > t|\mathcal{F}_t^{(N)});$$

we shall also use the following notation:

$$\mu_t(h_t) = \lambda_1(k)P(U \leq t \,|\, h_t) + \lambda_0(k)P(U > t \,|\, h_t). \tag{3}$$

A counting process that is a Pure Birth process conditionally on the change point, as described so far, will be denoted by the symbol CPB$(G, \lambda_0(\cdot), \lambda_1(\cdot))$.

Assume now

$$\lambda_1(k) \geq \lambda_0(k) \text{ for } k = 0, 1, \ldots \tag{4}$$

and compare two different observed histories

$$h'_t \equiv \{T_1 = t'_1, \ldots, T_k = t'_k, T_{k+1} > t\} \tag{5}$$

$$h''_t \equiv \{T_1 = t''_1, \ldots, T_k = t''_k, T_{k+1} > t\} \tag{6}$$

both containing k arrivals in the same time-interval $[0, t]$.

We write $h''_t \trianglerighteq h'_t$ if

$$t''_i \geq t'_i \text{ for } i = 1, 2, \ldots, k \tag{7}$$

or, equivalently, for any $s \in [0, t)$,

$$\sum_{j=1}^{\infty} 1_{\{s\leq t''_j \leq t\}} = \sum_{j=1}^{k} 1_{\{s\leq t''_j \leq t\}} \geq \sum_{j=1}^{k} 1_{\{s\leq t'_j \leq t\}} = \sum_{j=1}^{\infty} 1_{\{s\leq t'_j \leq t\}} \tag{8}$$

i.e. $h''_t \trianglerighteq h'_t$ when the number of recent arrivals in the history h'_t is not larger than the number of recent arrivals in h''_t.

In the present note we analyze some aspects of CPB counting processes and, in particular, we prove the following result.

Theorem 1. *If $h''_t \trianglerighteq h'_t$ then $\mu_t(h''_t) \geq \mu_t(h'_t)$.*

Our interest for this result is illustrated in the following remark.

Remark 1.1. *When U is exponentially distributed, the computation of $\mu_t(h_t)$ can be in principle carried out explicitly; in fact, in such a case, one can compute the normalizing constant that is needed to obtain the conditional probabilities in Eq. (3) and this in turn allows Theorem 1 to admit a direct proof. The case with $\lambda_1(0) = \lambda_1(1) = \cdots$ and $\lambda_0(0) = \lambda_0(1) = \cdots$ is dealt with in [7]; more lengthy expressions may be involved in our case where λ_0 and λ_1 may depend on the number of past arrivals. The explicit computation of the normalizing constant in Eq. (3) is however not possible when U is not exponentially distributed.*

Obviously $\{N_t\}_{t\geq 0}$ is not a pure birth-process (i.e. it is not Markov): when we "uncondition" with respect to the random variable $\mathbf{1}_{\{U\leq t\}}$ in Eq. (3), we obtain an intensity $\mu_t(h_t)$ which depends on the arrival times $t_1, t_2, \ldots, t_k$ and not only on k. It is then natural to wonder whether it is possible to establish some a priori inequalities on the pair $(\mu_t(h'_t), \mu_t(h''_t))$.

The paper will be organized as follows. In Sect. 2 we will consider a random change of time-scale that will reveal to be useful in the proof of Theorem 1; we will in particular show that the class of CPB processes is closed under this type of transformation.

Theorem 1 will be proved in Section 3. On this purpose, we prove an analogous result for a corresponding discrete-time model, afterwards the desired result will be obtained by means of a suitable passage to the limit. We notice that the discrete time result can be however of autonomous interest.

Section 4 will be devoted to a brief discussion and to some final remarks on Theorem 1 and on the class of CPB counting processes. Models in this class emerge in a natural way in several fields; in particular we shall mention two cases of interest, in the frame of reliability and experimental sciences, respectively.

For several aspects of the well-known change-point problem and a comprehensive bibliography, we address the reader *e.g.* to [1, 6, 7, 10] and references therein; we refer to Bremaud [4, 5] for general aspects about counting processes. For properties of monotonicity and of stochastic orderings for counting processes, see [9] and [11].

2 A random time-scale transformation

Besides the process $\{N_t\}_{t\geq 0}$, we shall introduce in this section a new counting process $\{\tilde{N}_t\}_{t\geq 0}$; Lemma 2.1 and Proposition 2.1 to be obtained below will turn out to be useful for our purposes in the next section. Lemma 2.1 in particular shows that the conditional probability of the event $\{U > t\}$, given an observed history h_t for $\{N_t\}_{t\geq 0}$, does coincide with an analogous conditional probability for $\{\tilde{N}_t\}_{t\geq 0}$.

Such a new process, which also admits intensities, is obtained from the original one by means of a random change of time-scale, as follows.

Let $(\Omega, \mathcal{F}, P)$ be the probability space on which the random variables $U, T_1, T_2, \dots$ are defined and let $\gamma_0, \gamma_1, \dots$ be a sequence of positive constants with

$$0 < \inf_i \gamma_i \leq \sup_i \gamma_i < \infty.$$

Let $g : \Omega \times [0, \infty) \to [0, \infty)$ be the strictly increasing random function of time defined as follows

$$g(\omega, t) = \sum_{k=0}^{N_t(\omega)-1} \gamma_k (T_{k+1}(\omega) - T_k(\omega)) + \gamma_{N_t(\omega)}(t - T_{N_t(\omega)}(\omega)). \tag{9}$$

¿From now on the symbol ω will be dropped; using a more compact notation we write

$$g(t) = \int_0^t \sum_{k=0}^{\infty} \gamma_k 1_{[T_k, T_{k+1})}(s) ds,$$

or, by setting

$$\gamma(s) = \sum_{k=0}^{\infty} \gamma_k 1_{[T_k, T_{k+1})}(s),$$

$$g(t) = \int_0^t \gamma(s) ds.$$

Define now, on $(\Omega, \mathcal{F}, P)$, the random variables

$$\tilde{U} = g(U) = \sum_{k=0}^{N_U - 1} \gamma_k (T_{k+1} - T_k) + \gamma_{N_U}(U - T_{N_U}) = \int_0^U \gamma(s) ds \tag{10}$$

$$\tilde{T}_l = g(T_l) = \sum_{k=0}^{l-1} \gamma_k (T_{k+1} - T_k) = \int_0^{T_l} \gamma(s) ds \text{ for } l = 1, 2, \dots. \tag{11}$$

and consider the new counting process $\{\tilde{N}_t\}_{t \geq 0}$ whose arrival times are $\tilde{T}_1$, $\tilde{T}_2, \dots$; thus we have

$$\tilde{N}_{g(t)} = N_t. \tag{12}$$

Let $A_1, A_2, ...$ denote the *interarrival times* of $\{N_t\}_{t \geq 0}$:

$$N_t = \sup\{n | \sum_{k=1}^{n} A_k \leq t\}.$$

Notice that the transformation yielding $\{\widetilde{N}_t\}_{t \geq 0}$ can also be described by writing

$$\widetilde{N}_t = \sup\{n | \sum_{k=1}^{n} \gamma_{k-1} A_k \leq t\},$$

i.e. $\{\widetilde{N}_t\}_{t\geq 0}$ is such that its interarrival times $\widetilde{A}_1,\ \widetilde{A}_2,\ldots,$ satisfy

$$\widetilde{A}_k = \gamma_{k-1} A_k. \tag{13}$$

We denote $\tilde{X}_t \equiv 1_{\{\tilde{U}<t\}}$, and consider the filtrations $\tilde{\Im} \equiv \{\mathcal{F}_t^{(\tilde{N})}\}_{t\geq 0}$, $\tilde{\aleph} \equiv \{\mathcal{F}_t^{(\tilde{N},\tilde{X})}\}_{t\geq 0}$. From now on, for typographic convenience, we shall often use the symbols $N(t)$ and $\tilde{N}(t)$ in place of N_t and $\tilde{N}_t$, respectively.

As we shall see, the interest in the transformation defined by (9), is motivated by the following Lemma.

Lemma 2.1. *Under the positions (9), (10), and (11) one has*

a) $\mathcal{F}_{g(t)}^{(\tilde{N},\tilde{X})} = \mathcal{F}_t^{(N,X)}$ *and* $\mathcal{F}_{g(t)}^{(\tilde{N})} = \mathcal{F}_t^{(N)}$;

b) $P(U > t|\mathcal{F}_t^{(N)}) = P(\tilde{U} > g(t)|\mathcal{F}_{g(t)}^{(\tilde{N})})$.

Proof. First we notice the following: since the transformation g defined by (9) is continuous and increasing in t, we have

$$\{\omega\in\Omega : N_t = k, T_1\in[t_1, t_1+\Delta_1), \ldots, T_k\in[t_k, t_k+\Delta_k), T_{k+1}>t, U>s\} =$$

$$\big\{\omega\in\Omega : \tilde{N}_{g(t)} = k, \tilde{T}_1\in\big[g(t_1), g(t_1{+}\Delta_1)\big), \ldots, \tilde{T}_k\in\big[g(t_k), g(t_k{+}\Delta_k)\big), \tilde{T}_{k+1}>g(t), \tilde{U}>g(s)\big\}, \tag{14}$$

and

$$\big\{\omega\in\Omega : N_t = k, T_1\in\big[t_1, t_1+\Delta_1\big), \ldots, T_k\in\big[t_k, t_k+\Delta_k\big), T_{k+1}>t\big\} =$$

$$= \big\{\omega\in\Omega : \tilde{N}_{g(t)} = k, \tilde{T}_1\in\big[g(t_1), g(t_1+\Delta_1)\big), \ldots, \tilde{T}_k\in\big[g(t_k), g(t_k+\Delta_k)\big), \tilde{T}_{k+1}>g(t)\big\}. \tag{15}$$

a) $\mathcal{F}_t^{(N,X)}$ is actually generated by the subsets of the type

$$\{\omega\in\Omega : N_t = k, T_1\in[t_1, t_1+\Delta_1), \ldots, T_k\in[t_k, t_k+\Delta_k), T_{k+1}>t, U>s\}$$

with $k = 0, 1, \ldots,$ $s\leq t$, $0\leq t_1\leq\cdots\leq t_k\leq t$,
and $\mathcal{F}_{g(t)}^{(\tilde{N},\tilde{X})}$ is generated by the subsets of the type

$$\big\{\omega\in\Omega : \tilde{N}_{g(t)} = k, \tilde{T}_1\in\big[g(t_1), g(t_1+\Delta_1)\big), \ldots, \tilde{T}_k\in\big[g(t_k), g(t_k+\Delta_k)\big), \tilde{T}_{k+1}>g(t), \tilde{U}>g(s)\big\}.$$

Then the identity $\mathcal{F}_{g(t)}^{(\tilde{N},\tilde{X})} = \mathcal{F}_t^{(N,X)}$ follows from (14). Similarly $\mathcal{F}_{g(t)}^{(\tilde{N})} = \mathcal{F}_t^{(N)}$ follows from (15).

b) The assertion immediately follows from a) by noticing that $\{U > t\} = \{\tilde{U} > g(t)\}$.

□

Remark 2.1. *The intensity $\mu_t(h_t)$ can be expressed in terms of the process $\{\widetilde{N}_t\}_{t\geq 0}$; in fact, by b) of Lemma 2.1 and by recalling the notation (3), we can write*

$$\mu_t(h_t) = \lambda_0(k) + (\lambda_1(k) - \lambda_0(k))P(\tilde{U} \leq g(t) \,|\, \tilde{T}_1 = g(t_1), \ldots, \tilde{T}_k = g(t_k), \tilde{T}_{k+1} > g(t)). \tag{16}$$

The class of CBP processes is closed under the transformation defined by (9); more precisely we have the following result that can also be inspired by equation (13).

Proposition 2.1. *The process $\{\tilde{N}_t\}_{t\geq 0}$ is CPB$(\tilde{G}, \tilde{\lambda}_1, \tilde{\lambda}_0)$ where $\tilde{G}(s) \equiv P(\tilde{U} < s)$ and*

$$\tilde{\lambda}_i(k) = \frac{\lambda_i(k)}{\gamma_k}. \tag{17}$$

Actually Proposition 2.1 could be proved by using a general, well known, result about simple counting processes; the latter shows how a simple counting process, admitting intensity, can be obtained from a standard Poisson process via a random change of time scale (see e.g.[5, 8]). For the reader's convenience, we prefer however to give here a direct proof which uses the specific notation of this paper.

The following remark will be used in such a proof.

Remark 2.2. *The events $\{\tilde{N}(g(t+\Delta)) > \tilde{N}(g(t))\}$ and $\{\tilde{N}(g(t) + \gamma_{\tilde{N}(g(t))}\Delta) > \tilde{N}(g(t))\}$ are equal. In order to show such identity we notice:*

$$\{N(t+\Delta) > N(t)\} = \{\tilde{N}(g(t+\Delta)) > \tilde{N}(g(t))\} \tag{18}$$

furthermore, if there is no arrival in the interval $(t, t+\Delta]$ for the original process N, we can write

$$g(t+\Delta) = g(t) + \gamma_{\tilde{N}(g(t))}\Delta = g(t) + \gamma_{N(t)}\Delta, \tag{19}$$

whence $\tilde{N}(g(t+\Delta)) = \tilde{N}(g(t) + \gamma_{N(t)}\Delta)$, i.e. we can conclude that, if $N(t+\Delta) = N(t)$ then $\tilde{N}(g(t) + \gamma_{N(t)}\Delta) = \tilde{N}(g(t))$ as well.

Let us suppose, on the other hand, that there are one or more arrivals for the process N in the interval $(t, t+\Delta]$ and denote by T_a the instant of the earliest among these such arrivals. Then there is an arrival for $\tilde{N}$ at the instant $g(T_a)$, that is within the interval $(g(t), g(T_a)] = (g(t), g(t) + \gamma_{\tilde{N}(g(t))}(T_a - t)]$. This means that there is at least one arrival in $(g(t), g(t)\gamma_{\tilde{N}(g(t))}\Delta]$; in fact $(g(t), g(t)\gamma_{\tilde{N}(g(t))}\Delta] \supseteq (g(t), g(t) + \gamma_{\tilde{N}(g(t))}(T_a - t)]$, since $\Delta \geq (T_a - t)$.

Proof of Proposition 2.1. We know that

$$\lim_{\Delta\to 0^+} \frac{P(N(t+\Delta) - N(t) > 0 | \mathcal{F}_t^{(N,X)})}{\Delta} = \lambda_{X(t)}(N_t), \tag{20}$$

and, taking into account Lemma 2.1 and Eq. (12),

$$\lim_{\Delta\to 0^+} \frac{P(N(t+\Delta) - N(t) > 0 | \mathcal{F}_t^{(N,X)})}{\Delta} = \lim_{\Delta\to 0^+} \frac{P(\tilde{N}(g(t+\Delta)) - \tilde{N}(g(t)) > 0 | \mathcal{F}_{g(t)}^{(\tilde{N},\tilde{X})})}{\Delta}. \tag{21}$$

On the other hand in view of the previous Remark 2.2 we have

$$\lim_{\Delta\to 0^+}\frac{P(\tilde N(g(t+\Delta))-\tilde N(g(t))>0|\mathcal{F}^{(\tilde N,\tilde X)}_{g(t)})}{\Delta}=\lim_{\Delta\to 0^+}\frac{P(\tilde N(g(t)+\gamma_{\tilde N(g(t))}\Delta)-\tilde N(g(t))>0|\mathcal{F}^{(\tilde N,\tilde X)}_{g(t)})}{\Delta}$$

$$=\lim_{\Delta\to 0^+}\frac{P(\tilde N(g(t)+\gamma_{\tilde N(g(t))}\Delta)-\tilde N(g(t))>0|\mathcal{F}^{(\tilde N,\tilde X)}_{g(t)})}{\gamma_{\tilde N(g(t))}\Delta}\frac{\gamma_{\tilde N(g(t))}\Delta}{\Delta}$$

$$=\gamma_{\tilde N(g(t))}\lim_{\epsilon\to 0^+}\frac{P(\tilde N(g(t)+\epsilon)-\tilde N(g(t))>0|\mathcal{F}^{(\tilde N,\tilde X)}_{g(t)})}{\epsilon}.$$

Then, by (20), we can write

$$\lim_{\epsilon\to 0^+}\frac{P(\tilde N(g(t)+\epsilon)-\tilde N(g(t))>0|\mathcal{F}^{(\tilde N,\tilde X)}_{g(t)})}{\epsilon}=\frac{\lambda_{X(t)}(N_t)}{\gamma_{\tilde N(g(t))}}=\frac{\lambda_{X(t)}(N_t)}{\gamma_{N(t)}}$$

i.e.

$$\tilde\lambda_{\tilde X(g(t))}(\tilde N_{g(t)})=\lim_{\epsilon\to 0^+}\frac{P(\tilde N(g(t)+\epsilon)-\tilde N(g(t))>0|\mathcal{F}^{(\tilde N,\tilde X)}_{g(t)})}{\epsilon}=\frac{\lambda_{X(t)}(N_t)}{\gamma_{N(t)}}.$$

Thus for $i=0,1$ and for every $k=0,1,\dots$ we obtain

$$\tilde\lambda_i(k)=\frac{\lambda_i(k)}{\gamma_k}.$$

This completes the proof. □

3 Discrete approximations and proof of Theorem 1

We will start this section by considering a discrete approximation of the continuous-time model; this will allow us to prove a discrete-time version of Theorem 1 under an additional condition (see (28) below).

Afterwards, by performing a natural limit, we will obtain the desired result for the continuous-time model. In order to eliminate the condition (28) we shall resort to the counting process $\{\tilde N_t\}_{t\geq 0}$ and to the related results obtained in the previous section.

Consider a discrete-time model defined as follows. Let $\bar U$ be an $\mathbb{N}$-valued random time and set, for $m=1,2,\dots$

$$\bar X_n\equiv 1_{\{\bar U<n\}},\ \ \nu(m)\equiv P(\bar U=m|\bar U>m-1), \tag{22}$$

so that

$$P(\bar U=1)=\nu(1),\quad P(\bar U=m)=\nu(m)\prod_{l=1}^{m-1}(1-\nu(l)), \tag{23}$$

and we assume

$$\nu(m) \in (0,1). \tag{24}$$

Let $\bar{T}_1 < \bar{T}_2 < \dots$ be an increasing sequence of $\mathbb{N}$-valued random times and set

$$\bar{N}_n \equiv \sup\{k | \bar{T}_k \leq n\}. \tag{25}$$

We assume that two sequences of positives constants $\{\bar{\lambda}_0(k)\}_{k=0,1,\dots}$ and $\{\bar{\lambda}_1(k)\}_{k=0,1,\dots}$ exist such that

$$\begin{aligned} P(\bar{T}_{k+1} = n+1 | \bar{T}_1 = n_1, \dots, \bar{T}_k = n_k, \bar{T}_{k+1} > n, \bar{U} > n) = \bar{\lambda}_0(k) \\ P(\bar{T}_{k+1} = n+1 | \bar{T}_1 = n_1, \dots, \bar{T}_k = n_k, \bar{T}_{k+1} > n, \bar{U} \leq n) = \bar{\lambda}_1(k) \end{aligned} \tag{26}$$

and that, for any $k \in \mathbb{N}$

$$\bar{\lambda}_1(k) > \bar{\lambda}_0(k). \tag{27}$$

Furthermore we assume here

$$\frac{(1-\bar{\lambda}_1(k-1))(1-\bar{\lambda}_0(k))}{(1-\bar{\lambda}_0(k-1))(1-\bar{\lambda}_1(k))} \geq 1 \text{ for } k = 1, \dots. \tag{28}$$

For an history $\bar{h}_n \equiv \{\bar{T}_1 = n_1, \dots, \bar{T}_k = n_k, \bar{T}_{k+1} > n\}$, (where $0 \leq n_1 < \dots < n_k \leq n$) we set

$$\bar{\mu}_n(\bar{h}_n) \equiv P(\bar{T}_{k+1} = n+1 | \bar{h}_n) = \bar{\lambda}_1(k) P(\bar{U} \leq n \,|\, \bar{h}_n) + \bar{\lambda}_0(k) P(\bar{U} > n \,|\, \bar{h}_n).$$

In Proposition 3.1 below we will use the following observation

Remark 3.1. *Let A, B, C, D, α, β, γ be positive constants and define*

$$\theta = \frac{C}{A+B+C+D}, \tag{29}$$

$$\theta' = \frac{C\gamma}{A\alpha + B\gamma + C\gamma + D\delta}. \tag{30}$$

If $\alpha/\gamma \geq 1$ and $\delta/\gamma \geq 1$ then $\theta \geq \theta'$.

Proposition 3.1. *Under the conditions (24), (27) and (28), we have*

$$\bar{\mu}_n(\bar{h}''_n) \geq \bar{\mu}_n(\bar{h}'_n) \tag{31}$$

for any pair $\bar{h}'_n$, $\bar{h}''_n$ such that $\bar{h}''_n \trianglerighteq \bar{h}'_n$.

Proof. First we notice that the inequality (31) is equivalent to

$$P(\bar{U} > n \mid \bar{h}''_n) \leq P(\bar{U} > n \mid \bar{h}'_n). \tag{32}$$

Now we denote, for $1 \leq n_1 < n_2 < \dots < n_k \leq n$ and $j = 1, 2, \dots$

$$g_j(n_1, n_2, \dots, n_k; n) = P(\bar{U} = j, \bar{T}_1 = n_1, \dots, \bar{T}_k = n_k, \bar{T}_{k+1} > n),$$

so that

$$P(\bar{U} > n \mid \bar{h}_n) = \frac{\sum_{j=n+1}^{\infty} g_j(n_1, n_2, \ldots, n_k; n)}{\sum_{j=1}^{\infty} g_j(n_1, n_2, \ldots, n_k; n)}. \tag{33}$$

In view of (23) and (26),

$$g_j(n_1, n_2, \ldots, n_k; n) =$$

$$\left[\nu(j) \prod_{r=1}^{j-1} (1 - \nu(r))\right] \prod_{i=0}^{k-1} \left\{ \bar{\lambda}_{\mathbf{1}(n_{i+1} > j)}(i) \prod_{r=n_i+1}^{n_{i+1}-1} [1 - \bar{\lambda}_{\mathbf{1}(r>j)}(i)] \right\} \prod_{r=n_k+1}^{n} [1 - \bar{\lambda}_{\mathbf{1}(r>j)}(i)] \tag{34}$$

where we have set $n_0 = 0$.

Now, on the space of possible "discrete" histories, let us define the operators Φ_i as follows. For an history $\bar{h}_n \equiv \{\bar{T}_1 = n_1, \ldots, \bar{T}_i = n_i, \ldots, \bar{T}_k = n_k, \bar{T}_{k+1} > n\}$, let

$$\Phi_i(\bar{h}_n) \equiv \{\bar{T}_1 = n_1, \ldots, \bar{T}_i = n_i + 1, \ldots, \bar{T}_k = n_k, \bar{T}_{k+1} > n\}$$

for i such that $i = 1, \ldots, k-1$ and $n_{i+1} > n_i + 1$,

$$\Phi_k(\bar{h}_n) \equiv \{\bar{T}_1 = n_1, \ldots, \bar{T}_k = n_k + 1, \bar{T}_{k+1} > n\} \quad \text{if } n_k < n,$$

and

$$\Phi_i(\bar{h}_n) \equiv \bar{h}_n$$

otherwise.

It is easy to check that any history $\bar{h}''_n$ such that $\bar{h}''_n \trianglerighteq \bar{h}'_n$ can be obtained from $\bar{h}'_n$, by applying the operators Φ_i a finite number of times.

Then we can reduce ourselves to show the validity of the inequality

$$P(\bar{U} > n \mid \bar{h}_n) \geq P(\bar{U} > n \mid \Phi_l(\bar{h}_n)) \text{ for } l \in \{1, 2 \ldots, k\}. \tag{35}$$

With l as in (35) we now let

$$A(n_1, \ldots, n_k; n) = \sum_{j=1}^{n_l - 1} g_j(n_1, \ldots, n_k; n),$$

$$B(n_1, \ldots, n_k; n) = \sum_{j=n_l+1}^{n} g_j(n_1, \ldots, n_k; n),$$

$$C(n_1, \ldots, n_k; n) = \sum_{j=n+1}^{\infty} g_j(n_1, \ldots, n_k; n).$$

Then we can rewrite formula (33) as

$$P(\bar{U} > n \mid \bar{h}_n) = \frac{C(n_1, \ldots, n_k; n)}{A(n_1, \ldots, n_k; n) + g_{n_l}(n_1, \ldots, n_k; n) + B(n_1, \ldots, n_k; n) + C(n_1, \ldots, n_k; n)}.$$

We now switch to obtaining the expression of $P(\bar{U} > n \mid \Phi_l(\bar{h}_n))$ in terms of $A(n_1,\ldots,n_k;n)$, $B(n_1,\ldots,n_k;n)$, $C(n_1,\ldots,n_k;n)$ and $g_{n_l}(n_1,\ldots,n_k;n)$.

Let us denote

$$\begin{gathered}
\hat{A}(n_1,\ldots,n_k;n) = \textstyle\sum_{j=1}^{n_l-1} g_j(n_1,\ldots,n_l+1,\ldots,n_k;n),\\
\hat{B}(n_1,\ldots,n_k;n) = \textstyle\sum_{k=n_l+1}^{n} g_k(n_1,\ldots,n_l+1,\ldots,n_k;n),\\
\hat{C}(n_1,\ldots,n_k;n) = \textstyle\sum_{k=n+1}^{\infty} g_k(n_1,\ldots,n_l+1,\ldots,n_k;n),\\
\hat{g} = g_{n_l}(n_1,\ldots,n_l+1,\ldots,n_k;n).
\end{gathered}$$

The following identities hold:

$$\hat{A}(n_1,\ldots,n_k;n) = \alpha \cdot A(n_1,\ldots,n_k;n), \tag{36}$$

$$\hat{B}(n_1,\ldots,n_k;n) = \gamma \cdot B(n_1,\ldots,n_k;n), \tag{37}$$

$$\hat{C}(n_1,\ldots,n_k;n) = \gamma \cdot C(n_1,\ldots,n_k;n), \tag{38}$$

$$\hat{g} = g_{n_l}(n_1,\ldots,n_l+1,\ldots,n_k;n) = \delta \cdot g_{n_l}(n_1,\ldots,n_l,\ldots,n_k;n), \tag{39}$$

with

$$\alpha = \frac{1-\bar{\lambda}_1(l-1)}{1-\bar{\lambda}_1(l)},\ \delta = \frac{[1-\bar{\lambda}_0(l-1)]\bar{\lambda}_1(l-1)}{\bar{\lambda}_0(l-1)[1-\bar{\lambda}_1(l)]},\ \gamma = \frac{1-\bar{\lambda}_0(l-1)}{1-\bar{\lambda}_0(l)}.$$

In order to check the validity of the identity (36), we can just notice that, for $1 \le j \le n_l - 1$, it is

$$g_j(n_1,\ldots,n_l+1,\ldots,n_k;n) = \frac{1-\bar{\lambda}_1(l-1)}{1-\bar{\lambda}_1(l)} g_j(n_1,\ldots,n_l,\ldots,n_k;n),$$

in view of formula (34); then

$$\hat{A}(n_1,\ldots,n_k;n) = \sum_{j=1}^{n_l-1} g_j(n_1,\ldots,n_l+1,\ldots,n_k;n) =$$

$$= \frac{1-\bar{\lambda}_1(l-1)}{1-\bar{\lambda}_1(l)} \sum_{j=1}^{n_l-1} g_j(n_1,\ldots,n_l,\ldots,n_k;n) = \alpha \cdot A(n_1,\ldots,n_k;n).$$

The validity of the identities (37)-(39) can be obtained in an analogous way with $j > n_l$ or $j = n_l$, respectively.

By the definitions of $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{g}$ and by taking into account the identities (36)-(39), we can now write

$$P(\bar{U} > n \mid \Phi_l(\bar{h}_n)) = \frac{\hat{C}}{\hat{A}+\hat{g}+\hat{B}+\hat{C}} = \frac{\gamma C}{\alpha A + \delta g + \gamma B + \gamma C} = \frac{C}{\frac{\alpha}{\gamma}A + \frac{\delta}{\gamma}g + B + C}.$$

It is immediately seen (by using Remark 3.1) that, in view of the assumptions (27)-(28), it is

$$P(\bar{U} > n \mid \Phi_l(\bar{h}_n)) \leq P(\bar{U} > n \mid \bar{h}_n)$$

and this proves the assertion. $\square$

We are now in a position to prove Theorem 1.

Proof. Let us assume for the moment that, besides the condition (4), also the following condition holds:

$$\lambda_1(k) - \lambda_0(k) < \lambda_1(k+1) - \lambda_0(k+1) \text{ for } k = 0, 1 \dots \tag{40}$$

Consider a sequence of discrete-time models as follows: for $m = 1, 2, , \dots$ let $\bar{U}^{(m)} \equiv \frac{[mU]}{m}$ and $\bar{T}_l^{(m)}$ $(l = 1, 2 \dots)$ be discrete random variables taking values on the set $\{0, \frac{1}{m}, \frac{2}{m}, \dots\}$ and such that

$$\bar{\lambda}_0^{(m)}(k) \equiv \frac{\lambda_0(k)}{m}, \quad \bar{\lambda}_1^{(m)}(k) \equiv \frac{\lambda_1(k)}{m}, \tag{41}$$

where we set

$$\begin{array}{l} \bar{\lambda}_0^{(m)}(k) \equiv P(\bar{T}_{k+1}^{(m)} = \frac{n+1}{m} | \bar{T}_1^{(m)} = \frac{n_1}{m}, \dots, \bar{T}_k^{(m)} = \frac{n_k}{m}, \bar{T}_{k+1}^{(m)} > \frac{n}{m}, \bar{U} > \frac{n}{m}), \\ \bar{\lambda}_1^{(m)}(k) \equiv P(\bar{T}_{k+1}^{(m)} = \frac{n+1}{m} | \bar{T}_1^{(m)} = \frac{n_1}{m}, \dots, \bar{T}_k^{(m)} = \frac{n_k}{m}, \bar{T}_{k+1}^{(m)} > \frac{n}{m}, \bar{U} \leq \frac{n}{m}). \end{array} \tag{42}$$

It can be checked that, for $h_t = \{T_1 = t_1, \dots, T_k = t_k, T_{k+1} > t\}$

$$P(U > t | h_t) = \lim_{m \to \infty} P\left(\bar{U}^{(m)} > \frac{[tm]}{m} \middle| \bar{T}_1 = \frac{[t_1 m]}{m}, \dots, \bar{T}_{k+1} > \frac{[tm]}{m} \right). \tag{43}$$

We do not report all the details; we limit ourselves to mention that, in order to obtain the identity (43), one has first to take into account

$$P(U > t | h_t) = \lim_{\Delta \to 0} \frac{P(U > t, T_1 \in [t_1, t_1 + \Delta), \dots, T_k \in [t_k, t_k + \Delta), T_{k+1} > t)}{P(T_1 \in [t_1, t_1 + \Delta), \dots, T_k \in [t_k, t_k + \Delta), T_{k+1} > t)}. \tag{44}$$

The r.h.s. of (44) can be shown to be equal to

$$\lim_{m \to \infty} \frac{\sum_{n=0}^{\infty} P(T_1 \in [t_1, t_1 + \Delta_m), \dots, T_k \in [t_k, t_k + \Delta_m | A_n^{(m)}(t))) P(A_n^{(m)}(t))}{\sum_{n=0}^{\infty} P(T_1 \in [t_1, t_1 + \Delta_m), \dots, T_k \in [t_k, t_k + \Delta_m | A_n^{(m)}(0))) P(A_n^{(m)}(0))}$$

where we have denoted, for $s \geq 0$, $A_n^{(m)}(s) = \{\bar{U}^{(m)} \in [s + n\Delta_m, s + (n+1)\Delta_m)\}$ and Δ_m is an infinitesimal sequence.

Finally the identity (43) can be obtained by a Poisson-type approximation by taking into account the position (41); for a general discussion about Poisson approximations and for results similar to the one needed here see e.g. [3].

Assuming (4) and (40) we obtain the conditions (27)-(28) for the intensities $\bar{\lambda}_0^{(m)}(k)$ and $\bar{\lambda}_1^{(m)}(k)$, in fact the condition (27) is trivially verified for all the integer m and the condition (28) is easily obtained from (40) by a Taylor expansion, for large m. Consider

now h'_t and h''_t as in (5), (6) and, for m large enough, the corresponding histories in discrete time defined by

$$\bar{h}'_t = \{\bar{T}_1 = \tfrac{[t'_1 m]}{m}, \ldots, \bar{T}_{k+1} > \tfrac{[tm]}{m}\}$$
$$\bar{h}''_t = \{\bar{T}_1 = \tfrac{[t''_1 m]}{m}, \ldots, \bar{T}_{k+1} > \tfrac{[tm]}{m}\}.$$

If $h''_t \trianglerighteq h'_t$ then, it is

$$\frac{[t'_i m]}{m} \leq \frac{[t''_i m]}{m} \text{ for } i = 1, \ldots, k.. \tag{45}$$

In view of condition (45) we have the inequality (32) (see the proof of Proposition 3.1); by using (43), we can then obtain, for the continuous-time limit process $\{N_t\}_{t \geq 0}$ the inequality

$$P(U > t \mid h''_t) \leq P(U > t \mid h'_t). \tag{46}$$

whence $\mu_t(h''_t) \geq \mu_t(h'_t)$ immediately follows under the assumption (40).

We now show however that such an assumption is by no means restrictive. Suppose in fact that we deal with a CPB$(G, \lambda_1, \lambda_0)$ with λ_1, λ_0 not satisfying (40) and apply the transformation $g(t)$ in (9) with the specific choice

$$\gamma_k = c_k \frac{\lambda_1(k) - \lambda_0(k)}{\lambda_1(0) - \lambda_0(0)}.$$

Here c_k is an arbitrary decreasing sequence such that $c_0 = 1$ and $\lim_{k \to \infty} c_k > 0$. By Lemma 2.1 we thus obtain a new CPB$(\tilde{G}, \tilde{\lambda}_1, \tilde{\lambda}_0)$ process where

$$\tilde{\lambda}_i(k) = \frac{\lambda_i(k)}{\gamma_k} = \lambda_i(k) \frac{1}{c_k} \frac{\lambda_1(0) - \lambda_0(0)}{\lambda_1(k) - \lambda_0(k)}$$

so that the condition (40) is satisfied. Lemma 2.1 then shows that we were entitled to prove Theorem 1 for the CPB$(\tilde{G}, \tilde{\lambda}_1, \tilde{\lambda}_0)$. $\square$

Obviously the conditional probability $P(U < t | h_t)$ in (3) actually depends on the parameters $\lambda_i(k)$ ($i = 0, 1$ and $k = 0, 1, \ldots$); then it might be convenient to use the notation $P_{\{\lambda_i(\cdot)\}}(U < t | h_t)$.

Notice that the inequality in (4) has been taken in the strict sense; however, by using the continuity property of $P_{\{\lambda_i(k)\}}(U < t | h_t)$ with respect to the set of parameters $\lambda_i(\cdot)$, it can be easily checked that in Theorem 1 the inequality can be taken in the broad sense.

4 Discussion and concluding remarks.

The notion of CPB processes, as it has been described in the Introduction, is a very natural model, that can emerge in several fields of application; it can be used to formalize a number of possible situations, that, apart from the use of different languages, turn out to be substantially isomorphic one to the other. Here we give just two possible instances, taken from different fields of application.

Example *(A reliability application)*. A typical problem in reliability modelling is the description of stochastic dependence among lifetimes of components that are to operate simultaneously in a same environment; two simple models of dependence in this respect, are quite common in the reliability literature: the standard *change-point* model and the *load-sharing* model.

The standard change-point model can be described as follows: n components $C_1, \ldots, C_n$, that we assume to be identical for simplicity's sake, start operate simultaneously and go on working, each C_i until its own failure time W_i and with no physical interaction with the others. However $C_1, ..., C_n$ are imbedded in a same environment and it is the case that the environmental condition will suddenly change its state at a random time U (the change point); this creates a form of stochastic dependence among the failure times $W_1, ..., W_n$: conditionally on $\{U = u\}$, $W_1, ..., W_n$ are independent with a same failure rate coinciding with a given function $\rho_0(t)$ for $t < u$ and coinciding with a different failure rate function $\rho_1(t)$ for $t \geq u$.

The load-sharing model emerges instead when $C_1, ..., C_n$ share a same load or share a benefit from a same favorable external condition: this makes that, between two subsequent failure times $W_{(i)}$ and $W_{(i+1)}$, the components that survived $W_{(i)}$ act independently, with a failure rate function dependent on the overall number $(n - i)$ of surviving components and, possibly, on the calendar time.

This situation is described by the fact that the counting process $\{N_t\}_{t \geq 0}$ with

$$N_t = \sum_{i=1}^{n} 1_{\{T_i \leq t\}}$$

is Markov (possibly non-homogeneous), i.e. it is a pure death process; for more details on this aspect see e.g. [2] and [14]; for some examples and a wider list of references on the load-sharing model, see also [12], [13], [15].

The CPB models considered in the present paper arise as a natural superposition of standard change-point and load-sharing models, as described so far. In fact, conditionally on the change point U, the failure-times $W_1, \ldots, W_n$ are not independent, but rather they obey a common load-sharing model and the counting process $\{N_t\}_{t \geq 0}$ is a CPB process. This is of interest in that one may often have to handle sets of components that share the same load (or the same stress) and the latter can suddenly increase its level at an unpredictable instant. The question may arise in those cases whether, under a same number of observed failures within a time-instant t, we have to be more pessimistic with early failure times or with very recent failure times.

Theorem 1 gives a response to this question under the condition that in any case the hazard of surviving components becomes more severe after the change-point.

Example *(An application in Physics)*. Under a different language, the same superimposition of a change- point model and a load-sharing model, can be of interest in the field of experimental sciences. One can think for instance of n spins $C_1, ..., C_n$ embedded into a uniform magnetic field; initially all the spins are in the state -1 and each of them flips to its ground state $+1$ in a random time W_i. We assume that, at any time-instant,

the transition rate, beside being an increasing function of the intensity of the magnetic field, is influenced also by the number of already flipped spins.

Furthermore we think of the cases where the intensity of the magnetic field, at time 0, has the value B_0 and, at a random time-instant U flips to value B_1, with $B_1 > B_0$. The CPB model applies when the underlying magnetic field is not directly observable.

Of course the examples above concern the case of counting processes with a finite number of arrivals in the interval $[0, \infty)$; examples of interest also can be found for the case of infinite arrivals.

We now conclude the paper with a remark about the pair of histories to be compared.

In Theorem 1 we compared two histories observed on the same time-interval $[0, t]$ and containing the same number of arrivals k. Consider now two different histories h_t' and h_t'' on the same time-interval $[0, t]$ where h_t'' is obtained from h_t' by simply "adding" some arrivals. Under assumption (4), one may guess that the inequality $\mu_t(h_t'') \geq \mu_t(h_t')$ holds.

We notice on the contrary that this is not true, as the following simple example shows. Let $\{N_t\}_{t\geq 0}$ be a CPB$(G, \lambda_0(\cdot), \lambda_1(\cdot))$ process with

$$\overline{G}(t) = \exp\{-t\},$$

$$\lambda_0(0) = \lambda_0(1) = 1; \lambda_1(0) = 2, \lambda_1(1) = M \gg 2.$$

and simply consider the two histories

$$h_t' \equiv \{\text{no arrival in } [0, t]\}; \quad h_t'' \equiv \{T_1 = t_t, T_2 > t\}.$$

It is easy to check that $\mu_t(h_t'') < \mu_t(h_t')$.

Some more assumptions on the conditional birth rates are then needed in order to get the inequality $\mu_t(h_t'') \geq \mu_t(h_t')$.

Some considerations analogous to those above can be made concerning the comparison between two histories that contain the same number of arrivals but are observed over two different time-intervals: consider e.g. the two histories $h_{t'}'$ and $h_{t''}''$ be given by

$$h_{t'}' = \{ \text{ no arrivals in } [0, t']\}; \;\; h_{t''}'' = \{ \text{ no arrivals in } [0, t'']\}$$

with $t' < t''$. It is clear that with appropriate choice of G and of the rates $\lambda_i(0)$ $(i = 0, 1)$ we can have

$$P(U > t'|h_{t'}') \leq P(U > t''|h_{t''}'')$$

or

$$P(U > t'|h_{t'}') > P(U > t''|h_{t''}'').$$

We can then conclude that, by only assuming the condition (4), an inequality as in Theorem 1 cannot be obtained by comparing two histories if they are not observed on the same time-interval and do not contain the same number of arrivals.

References

[1] Arjas, E., Haara, P. and Norros, I. (1992). Filtering the histories of a partially observed marked point process. *Stochastic Process. Appl.*, **40**, 225-250.

[2] Arjas, E. and Norros, I. (1991). Order and martingale dynamics in multivariate life length models: a review. *Stochastic orders and decision under risk* (Hamburg, 1989), 7-24, IMS Lecture Notes Monogr. Ser., 19, Inst. Math. Statist., Hayward, CA, 1991.

[3] Barbour, A.D., Holst, L., and Janson, S. (1992). Poisson Approximation. Oxford Science Publications, Oxford.

[4] Bremaud, P. (1981). Point processes and queues, Springer-Verlag, New York-Berlin.

[5] Bremaud, P. (2002). An initiation to point process calculus. In http://lcavwww.epfl.ch∼bremaud/

[6] Change-point problems. Papers from the AMS-IMS-SIAM Summer Research Conference held at Mt. Holyoke College, South Hadley, MA, July 11–16, 1992. Edited by Edward Carlstein, Hans-Georg Müller and David Siegmund. Institute of Mathematical Statistics Lecture Notes—Monograph Series, 23. Institute of Mathematical Statistics, Hayward, CA, 1994.

[7] Herberts, T. and Jensen, U. (2002). Optimal detection of a change point in a point process for different observation schemes. Submitted for pubblication.

[8] Kurtz T. G. (1982). Representation and approximation of counting processes. Advances in filtering and optimal stochastic control. Lecture Notes in Control and Inform. Sci., **42**, Springer, 177–191.

[9] Kwiecinski A. and Szekli R. (1996). Some monotonicity and dependence properties of self-exciting point processes. *Ann. Appl. Probab.*, **6**, 1211-1231.

[10] Muliere, P. and Scarsini M. (1993). Some aspects of Change-Point Problems. *Reliability and Decision Making.* (Siena, 1990), Chapman & Hall. 273-285

[11] Müller, A., Stoyan, D. (2002). Comparison methods for stochastic models and risks. Wiley Series in Probability and Statistics.

[12] Ross, S. M. (1984) A model in which components failure rates depend only on the working set. *Nav. Res. Log. Quart.,* **31**, 297-300

[13] Shaked, M., Shanthikumar, J. G. (1991) Dynamic multivariate aging notions in reliability theory. *Stochastic Process. Appl.* **38**, no. 1, 85–97

[14] Shaked M., Spizzichino F. and Suter F. (2002). Nonhomogeneous birth processes and l_∞-spherical densities, with applications in reliability theory. *Probab. Engrg. Inform. Sci.*, **16**, no. 3, 271–288.

[15] Spizzichino, F. (2001). Subjective Probability Models for Lifetimes, CRC Press.